J. Fluid Mech. (1970), *vol.* 43, *part* 1, *pp.* 165–176
Printed in Great Britain

The diffraction of internal waves by a semi-infinite barrier

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(Received 8 October 1969)

We discuss the diffraction of internal waves by a semi-infinite vertical barrier in a uniformly rotating, stably stratified fluid of constant depth and Brunt–Väisälä frequency. N, For the frequency passband $f < \sigma < N$, where f and σ are respectively the inertial and wave frequencies, the presence of rotation gives rise to internal Kelvin waves which propagate without attenuation along the barrier. For the passband $N < \sigma < f$, however, the barrier generates waves which propagate without attenuation away from the barrier and which have amplitudes that fall off exponentially in the direction along the barrier.

1. Introduction

The diffraction of long gravity waves by a semi-infinite vertical barrier in a uniformly rotating system has been considered by Crease (1956) for the case of normal incidence and later by Chambers (1964), for arbitrary incidence. One of the remarkable results of their work was that the presence of rotation gives rise to a Kelvin wave which propagates without attenuation into the region behind the barrier. The direction of propagation of this wave is along the barrier and its amplitude falls off exponentially in the direction normal to the barrier. Further, for certain ranges of the wave frequency and incident angle, the amplitude at the barrier exceeds that of the incident wave field.

In this paper we consider an analogous diffraction problem for internal waves in a uniformly rotating, stably stratified fluid of constant depth and Brunt–Väisälä frequency. We show that the boundary-value problem for the horizontal spatial dependence of the nth mode of the diffracted wave field, ψ_n, is formally equivalent to that derived by Crease and Chambers for the spatial dependence of the diffracted long wave, ζ. However, because of the existence of two frequency passbands for internal waves, the solutions for ψ_n and ζ are not identical in form. For the case $f < \sigma < N$, where f, σ and N are respectively the inertial, wave and Brunt–Väisälä frequencies, the solutions are identical in form. But for the case $N < \sigma < f$, no internal Kelvin wave exists immediately behind the barrier; instead there arises a wave which travels normal to the barrier without attenuation and which has an amplitude that falls off exponentially in the direction along the barrier.

2. Formulation of boundary-value problem

We consider the time-dependent motion of a uniformly rotating, inviscid and incompressible fluid in the domain R: $-\infty < x,\ y < \infty,\ -h < z < 0$ exterior to the semi-infinite strip $y = 0,\ x < 0$. Here x, y, z form a right-handed Cartesian system with z as the vertical co-ordinate. The mean sea level is at $z = 0$ and the ocean bottom at $z = -h$, a constant. The linearized equations for the conservation of momentum, mass and volume are

$$\left.\begin{aligned} u_t - fv + p_x/\rho_0 &= 0,\\ v_t + fu + p_y/\rho_0 &= 0,\\ w_t + p_z/\rho_0 + \rho g/\rho_0 &= 0,\end{aligned}\right\} \tag{2.1}$$

$$\rho_t + \rho_{0z} w = 0, \tag{2.2}$$

$$u_x + v_y + w_z = 0. \tag{2.3}$$

Here u, v, w are the perturbation velocity components in the x, y, z directions respectively; p is the perturbed pressure; $\rho_0(z)$ and ρ are the mean (stable) and perturbed density fields respectively; f is the inertial frequency and g is the acceleration of gravity. Applying the Boussinesq approximation (see Phillips 1966, § 2.4), ρ_0 and ρ_{0z} are now treated as constants. On the boundaries of R we impose the conditions

$$\left.\begin{aligned} w = 0 \quad \text{on} \quad z = 0 \quad \text{and} \quad z = -h,\\ v = 0 \quad \text{on} \quad y = 0 \quad (x < 0).\end{aligned}\right\} \tag{2.4}$$

The boundary condition at $z = 0$ implies that there is no vertical motion at the sea surface; i.e. coupling between surface and internal waves is ignored in this analysis.

From (2.1) to (2.3) it follows that the vertical velocity component satisfies the equation

$$\Delta w_{tt} + f^2 w_{zz} + N^2 \Delta_H w = 0, \tag{2.5}$$

where $\Delta_H = \partial_x^2 + \partial_y^2$, $\Delta = \Delta_H + \partial_z^2$ and $N^2 = -g\rho_{0z}/\rho$ is the Brunt–Väisälä frequency, assumed to be constant. The horizontal velocity components are related to the vertical velocity component by the equations

$$\left.\begin{aligned} u_{zt} &= L(w_{xt} + f w_y),\\ v_{zt} &= L(w_{yt} - f w_x),\end{aligned}\right\} \tag{2.6}$$

where

$$L = (\partial_t^2 + f^2)^{-1}(\partial_t^2 + N^2).$$

Upon assuming a periodic time dependence

$$w(x,y,z,t) = W(x,y,z)\exp(-i\sigma t) \quad (\sigma > 0), \tag{2.7}$$

and defining

$$\omega^2 = (N^2 - \sigma^2)/(\sigma^2 - f^2), \tag{2.8}$$

equation (2.5) becomes

$$\omega^2 \Delta_H W - W_{zz} = 0. \tag{2.9}$$

From (2.4), (2.6) and (2.7) we find that the boundary conditions for W are

$$W = 0 \quad \text{on} \quad z = 0 \quad \text{and} \quad z = -h, \tag{2.10a}$$

$$W_y - i\gamma W_x = 0 \quad \text{on} \quad y = 0 \quad (x < 0), \tag{2.10b}$$

where $\gamma = f/\sigma$.

Let W_0 be a wave approaching the barrier at an angle $\theta(|\theta| < \pi)$ to the x axis (see figure 1). We assume W_0 has the form

$$W_0 = \sum_{n=1}^{\infty} s_n \exp\left[-ik_n(x\cos\theta + y\sin\theta)\right]\sin(n\pi z/h), \tag{2.11}$$

where $k_n^2 = (n\pi/\omega h)^2 > 0$; clearly W_0, being a superposition of normal modes, satisfies (2.9) and (2.10*a*). The coefficients s_n, which are assumed known, determine the shape of the incident wave. The condition $k_n^2 > 0$ together with (2.8) implies that two frequency passbands are physically realizable: $f < \sigma < N$ and

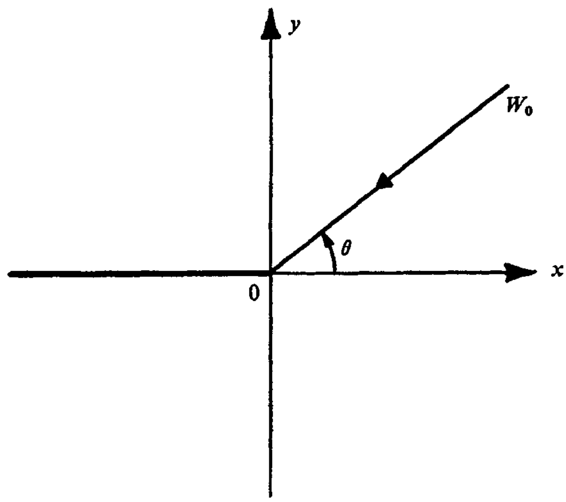

FIGURE 1. Plan view of wave approaching semi-infinite barrier $y = 0$, $x < 0$.

$N < \sigma < f$. Since the barrier gives rise to a diffracted wave field which must also satisfy (2.10*a*), we take the total wave field W to have the form

$$W = \sum_{n=1}^{\infty} s_n \psi_n^T(x, y)\sin(n\pi z/h), \tag{2.12}$$

where $\psi_n^T = \exp\left[-ik_n(x\cos\theta + y\sin\theta)\right] + \psi_n(x, y)$. Here $\psi_n(x, y)$ represents the horizontal spatial dependence of the nth mode of the diffracted wave field. The substitution of (2.12) into (2.9) and (2.10*b*) gives, upon dropping the subscript n,

$$(\Delta_H + k^2)\psi = 0, \tag{2.13a}$$

$$\psi_y(x, 0) - i\gamma\psi_x(x, 0) = a\exp(-ikx\cos\theta) \quad (x < 0), \tag{2.13b}$$

where $a = k(i\sin\theta + \gamma\cos\theta)$. To determine a unique solution to (2.13*a*, *b*) we also specify that (i) ψ satisfies the Sommerfeld radiation condition, (ii) ψ is bounded everywhere in R and $\psi_x(0+, 0)$ and $\psi_y(0+, 0)$ are integrable and (iii) ψ and $\psi_y - i\gamma\psi_x$ are continuous across $y = 0$ for $x > 0$ and all x respectively. Condition (iii), which ensures that w and v are continuous across $y = 0$ for $x > 0$ and all x respectively, also ensures that u is continuous across $y = 0$ for $x > 0$.

From (2.8) and (2.11) we see that the wave eigenmodes satisfy the dispersion relation

$$K^2 = (\sigma^2 - f^2)/(N^2 - \sigma^2), \tag{2.14}$$

where $K = kh/n\pi$. Equation (2.14) implies that $\sigma = \sigma(k)$, i.e. that σ depends upon only the magnitude of the horizontal wave-number vector (k_1, k_2). Thus, the wave group velocity components c_{gi} are given by

$$c_{gi} \equiv \partial\sigma/\partial k_i = (k_i/k)\, d\sigma/dk.$$

Now, the wave phase velocity components c_i are defined as

$$c_i \equiv k_i\,\sigma/k^2,$$

and hence, from (2.14), the scalar product of the phase and group velocities is

$$\mathbf{c}\,.\,\mathbf{c}_g = \frac{\sigma}{k}\frac{\partial\sigma}{\partial k} = \frac{h^2(N^2-f^2)}{(n\pi)^2\,(1+K^2)^2}. \tag{2.15}$$

This implies that whereas the phase and group velocities are in the same direction when $f < \sigma < N$, the group velocity has the opposite sign to that of the phase velocity when $N < \sigma < f$.† Thus, in order that the vector in figure 1 represents the direction of incoming wave energy, we take $k > 0$ for $f < \sigma < N$ and $k < 0$ for $N < \sigma < f$ in (2.11)–(2.13 a, b). Equation (2.15) must also be taken into consideration when applying the Sommerfeld radiation condition, namely, that the diffracted wave field ψ contains outward propagating energy only.

We note here that equations (2.13 a, b) with $k > 0$ also govern the behaviour of the diffracted wave due to a long gravity wave incident upon a semi-infinite vertical barrier in a rotating system (see Crease 1956, Chambers 1964). In that case ψ represents the spatial dependence of the sea surface elevation and $k^2 = (\sigma^2-f^2)/gh > 0$ is the wave-number of a freely propagating long wave. Further, for long waves in a rotating fluid it follows that $\mathbf{c}\,.\,\mathbf{c}_g = gh > 0$. Thus the two diffraction problems are mathematically equivalent when $\gamma < 1$ in the sense that both can be reduced to the same boundary-value problem (BVP) which in turn reduces to the BVP associated with the classical Sommerfeld diffraction problem when $\gamma = 0$ (no rotation).

3. Integral representation of solution for $f < \sigma < N$

To construct the solution to (2.13 a, b) (with $\theta = \frac{1}{2}\pi$), Crease (1956) first converts the BVP into an equivalent integral equation. From the latter he derives a Wiener–Hopf integral equation for a function $m(x)$ (see (3.5) below) whose kernel consists of the appropriate Green's function for the BVP. He then uses the Wiener–Hopf method to solve this second integral equation for $m(x)$. The substitution of $m(x)$ into the original integral equation then yields, after some manipulations, a Fourier integral for the diffracted wave. Crease's approach is unnecessarily complicated, however, since (2.13 a, b) can be solved directly by the Wiener–Hopf method without first recasting the BVP into an integral equation. We briefly outline this alternative approach (see Carrier *et al.* 1966, § 8.1) below.

Crease's problem (for arbitrary θ) has also been solved by Chambers (1964) who used a method previously developed by himself (Chambers 1954). Chambers assumes that the diffracted wave is equal to a linear combination of 'diffraction

† We are grateful to a referee for pointing out this fact.

functions', each of which is proportional to a certain complex Fresnel integral. The unknown coefficients in this equation are then determined by appropriate boundary conditions. While Chambers' approach is fairly elegant, it does not appear to be any simpler than the method we now give.

Upon applying to (2.13 a) the Fourier transform with respect to x as defined by

$$\overline{F}(\lambda, y) = \int_{-\infty}^{\infty} e^{i\lambda x} F(x,y)\,dx,$$

we obtain

$$\overline{\psi}_{yy} - (\lambda^2 - k^2)\,\overline{\psi} = 0. \tag{3.1}$$

The solution to (3.1) which is bounded as $|y| \to \infty$ is

$$\overline{\psi} = \begin{cases} A(\lambda)\exp\,[y(\lambda^2-k^2)^{\frac{1}{2}}] & (y<0), \\ B(\lambda)\exp\,[-y(\lambda^2-k^2)^{\frac{1}{2}}] & (y>0), \end{cases} \tag{3.2}$$

where the branch of $(\lambda^2-k^2)^{\frac{1}{2}}$ is chosen so that $\arg(\lambda^2-k^2)^{\frac{1}{2}} \to 0$ as $\lambda \to \pm\infty$. It follows that the radiation condition is satisfied if k is assumed to have a small

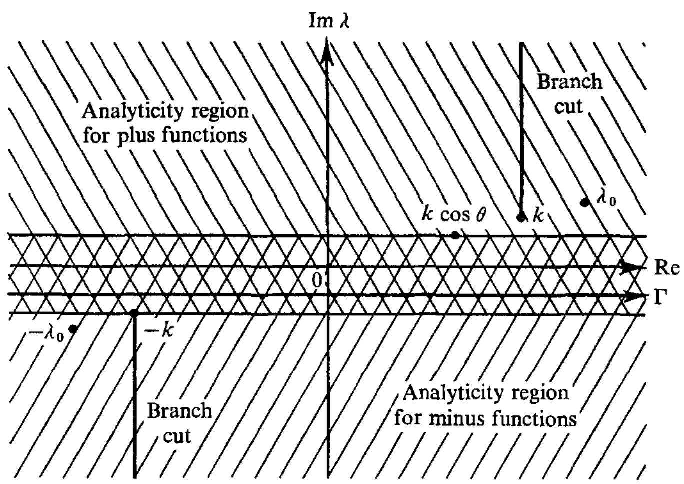

FIGURE 2. The λ plane corresponding to (3.10) for $f < \sigma < N$.

positive imaginary part, viz. $k = k_0 + i\epsilon$ ($k_0 > 0$, $\epsilon > 0$). Thus we draw the branch cuts from $\lambda = \pm k$ in the λ plane as shown in figure 2. The continuity of $\psi_y - i\gamma\psi_x$ across $y = 0$ implies

$$B(\lambda) = \left(\frac{\gamma\lambda - (\lambda^2-k^2)^{\frac{1}{2}}}{\gamma\lambda + (\lambda^2-k^2)^{\frac{1}{2}}}\right) A(\lambda). \tag{3.3}$$

To determine $A(\lambda)$, and hence $B(\lambda)$, we now introduce two half-known functions $g(x)$ and $m(x)$ by the equations

$$\psi_y(x,0) - i\gamma\psi_x(x,0) = \begin{cases} g(x) & (x>0), \\ a\exp(-ikx\cos\theta) & (x<0), \end{cases} \tag{3.4}$$

$$\psi(x,0+) - \psi(x,0-) = m(x), \tag{3.5}$$

where

$$g(x) = \begin{cases} ? & (x>0), \\ 0 & (x<0), \end{cases} \tag{3.6}$$

$$m(x) = \begin{cases} 0 & (x>0), \\ ? & (x<0). \end{cases} \tag{3.7}$$

Transforming (3.4) and using (3.2) (for $y < 0$) and (3.6) we obtain

$$[(\lambda^2-k^2)^{\frac{1}{2}}-\gamma\lambda]\,A(\lambda) = \left(\frac{-ia}{\lambda-k\cos\theta}\right)_- + \bar{g}_+(\lambda), \tag{3.8}$$

where the subscript $-$ denotes a minus function, which is analytic in the lower half plane (LHP) $\operatorname{Im}\lambda < \epsilon\cos\theta$ and the subscript $+$ denotes a plus function, which is assumed to be analytic in the UHP $\operatorname{Im}\lambda > -\epsilon$. Transforming (3.5) and using (3.2) and (3.7) we obtain

$$B(\lambda)-A(\lambda) = \overline{m}_-(\lambda), \tag{3.9}$$

where $\overline{m}_-$ is assumed to be analytic in the LHP $\operatorname{Im}\lambda < \epsilon$. Using (3.3) in (3.9) and substituting the result for A into (3.8), we obtain the Wiener–Hopf equation

$$\frac{[k^2-(1-\gamma^2)\,\lambda^2]\,\overline{m}(\lambda)}{2(\lambda^2-k^2)^{\frac{1}{2}}} = \frac{-ia}{\lambda-k\cos\theta}+\bar{g}(\lambda),$$

which can be rewritten in the form

$$\left[\frac{(\gamma^2-1)\,(\lambda-\lambda_0)\,\overline{m}(\lambda)}{2(\lambda-k)^{\frac{1}{2}}}+\frac{ia(k+k\cos\theta)^{\frac{1}{2}}}{(\lambda_0+k\cos\theta)\,(\lambda-k\cos\theta)}\right]_-$$

$$= \left[\frac{-ia[(\lambda+k)^{\frac{1}{2}}\,(\lambda_0+k\cos\theta)+(k+k\cos\theta)^{\frac{1}{2}}\,(\lambda+\lambda_0)]}{(\lambda+\lambda_0)\,(\lambda-k\cos\theta)\,(\lambda_0+k\cos\theta)}+\frac{(\lambda+k)^{\frac{1}{2}}\,\bar{g}(\lambda)}{(\lambda+\lambda_0)}\right]_+, \tag{3.10}$$

where $$\lambda_0 = k(1-\gamma^2)^{-\frac{1}{2}}.$$

The left-hand side of (3.10) is analytic for $\operatorname{Im}\lambda < \epsilon\cos\theta$, and the right-hand side is analytic for $\operatorname{Im}\lambda > -\epsilon$. Thus (3.10) holds in the strip $-\epsilon < \operatorname{Im}\lambda < \epsilon\cos\theta$ and defines an entire function $E(\lambda)$ which is analytic in the whole λ plane by analytic continuation. Since $\psi(0-,0)$ is bounded and $\psi_x(0+,0)$, $\psi_y(0+,0)$ are integrable, it follows that $\overline{m} = O(\lambda^{-1})$ as $|\lambda|\to\infty$ in the LHP and $\bar{g} = O(\lambda^{-\delta})$ ($\delta > 0$) as $|\lambda|\to\infty$ in the UHP respectively. Therefore each side of (3.10) tends to zero as $|\lambda|\to\infty$ in the strip, and, by Liouville's theorem, $E(\lambda) \equiv 0$. Thus, upon setting the left-hand side of (3.10) equal to zero, we obtain

$$\overline{m}(\lambda) = \frac{-2ia(\lambda-k)^{\frac{1}{2}}\,(k+k\cos\theta)^{\frac{1}{2}}}{(\gamma^2-1)\,(\lambda_0+k\cos\theta)\,(\lambda-\lambda_0)\,(\lambda-k\cos\theta)}. \tag{3.11}$$

Taking the inverse Fourier transform of (3.2) in which A and B are now determined from (3.3), (3.9) and (3.11), we obtain

$$\psi(x,y) = \frac{b}{2\pi}\int_\Gamma d\lambda\,\frac{\gamma\lambda-\operatorname{sgn}y(\lambda^2-k^2)^{\frac{1}{2}}}{(\lambda-\lambda_0)\,(\lambda-k\cos\theta)\,(\lambda+k)^{\frac{1}{2}}}\exp[-i\lambda x-|y|\,(\lambda^2-k^2)^{\frac{1}{2}}], \tag{3.12}$$

where $b = k(\sin\theta-i\gamma\cos\theta)\,(k+k\cos\theta)^{\frac{1}{2}}/(1-\gamma^2)\,(\lambda_0+k\cos\theta)$ and the inversion path Γ lies in the strip $-\epsilon < \operatorname{Im}\lambda < \epsilon\cos\theta$, as shown in figure 2. It can be verified that (3.12) indeed satisfies (2.13a) and the associated boundary conditions. Further, for $\theta = \frac{1}{2}\pi$, equation (3.12) reduces to Crease's Fourier integral representation of the solution for the diffracted wave.

4. Asymptotic solution for $f < \sigma < N$

To determine the general nature of the diffracted wave (3.12), we now consider the asymptotic form of ψ for large $kr = k(x^2+y^2)^{\frac{1}{2}}$. However, in contrast to Crease (1956) who first transforms the solution into complex Fresnel integrals before determining the asymptotic behaviour, we apply the method of steepest descent directly to (3.12). Letting $x = r\cos\phi$, $y = r\sin\phi$, where $-\pi < \phi < \pi$, and $\lambda = k\zeta$, equation (3.12) becomes

$$\psi(r,\theta) = \frac{(\sin\theta - i\gamma\cos\theta)(1+\cos\theta)^{\frac{1}{2}}}{2\pi(1-\gamma^2)(\zeta_0+\cos\theta)}\int_{\Gamma^*} d\zeta \exp\left[-kr(i\zeta\cos\phi + |\sin\phi|(\zeta^2-1)^{\frac{1}{2}})\right]\frac{[\gamma\zeta - \operatorname{sgn}\phi(\zeta^2-1)^{\frac{1}{2}}]}{(\zeta-\zeta_0)(\zeta-\cos\theta)(1+\zeta)^{\frac{1}{2}}}, \tag{4.1}$$

where $\zeta_0 = \lambda_0 k^{-1}$ and the path of integration Γ^* is as shown in figure 3. We have taken $\epsilon = 0$, because it has served its purpose of determining the correct contour.

From the argument of the exponential function in (4.1), we find that the saddle-point is located at $\zeta_s = -\cos\phi$ and that the corresponding path of steepest descent, Γ_s, is given by

$$\zeta_i = -\frac{(\zeta_r+\cos\phi)(\zeta_r\cos\phi+1)}{|\sin\phi|\,[(\zeta_r+\cos\phi)^2+\sin^2\phi]^{\frac{1}{2}}},$$

where ζ_r, ζ_i are respectively the real and imaginary parts of ζ. Thus, three cases for Γ_s are possible, viz. $\cos\phi > 0$, $\cos\phi = 0$ and $\cos\phi < 0$, as shown in figure 3. The contribution to (4.1) from the saddle-point is

$$\psi_s(r,\phi) = \frac{|\sin\phi|\,(i\sin\phi - \gamma\cos\phi)(\sin\theta - i\gamma\cos\theta)(1+\cos\theta)^{\frac{1}{2}}}{(1-\gamma^2)(\zeta_0+\cos\theta)(\zeta_0+\cos\phi)(\cos\phi+\cos\theta)(1-\cos\phi)^{\frac{1}{2}}} \times \frac{e^{i(kr-\frac{1}{4}\pi)}}{(2\pi kr)^{\frac{1}{2}}} + O\left[\frac{1}{(kr)^{\frac{3}{2}}}\right]. \tag{4.2}$$

Equation (4.2) is not valid, however, for $\cos\phi \simeq -\cos\theta$ or $\cos\phi \simeq 1$.

From figure 3 it is clear that the contour Γ^* can be deformed into Γ_s without capturing any poles provided that $\cos\phi > -\cos\theta$ and $\cos\phi > -1/\zeta_0$. Then, by Cauchy's theorem, (4.2) is the leading term of the asymptotic solution for ψ. But when $\cos\phi < -\cos\theta$ or $\cos\phi < -1/\zeta_0$, there are additional contributions from the poles at $\zeta = \cos\theta$ or $\zeta = \zeta_0$, respectively. Thus, as $kr \to \infty$, the total wave mode ψ^T can be written in the form

$$\begin{aligned}\psi^T = {} & \psi_s(r,\phi) + [H(\cos\phi+\cos\theta) + H(-\cos\phi-\cos\theta) \\ & \times H(\operatorname{sgn}\phi\operatorname{sgn}\theta)]\exp[-ik(x\cos\theta+y\sin\theta)] \\ & + \frac{(\sin\theta - i\gamma\cos\theta)^2}{(\gamma^2\cos^2\theta+\sin^2\theta)}H(-\cos\phi-\cos\theta)H(\operatorname{sgn}\phi\sin\theta) \\ & \times \exp[-ik(x\cos\theta - y\sin\theta)] \\ & + \frac{2(\zeta_0-1)^{\frac{1}{2}}(1+\cos\theta)^{\frac{1}{2}}}{(\gamma\cos\theta - i\sin\theta)}H(-y)H(\zeta_0+1/\cos\phi)\exp[\zeta_0 k(\gamma y - ix)],\end{aligned} \tag{4.3}$$

where $\zeta_0 = (1-\gamma^2)^{-\frac{1}{2}}$ and $H(x)$ is the Heaviside unit step function. The first term in (4.3) represents a modulated cylindrical wave propagating out from the origin; it is given explicitly by equation (4.2). The incoming free wave is given by the second term in (4.3). It exists in the regions I and II shown in figure 4.

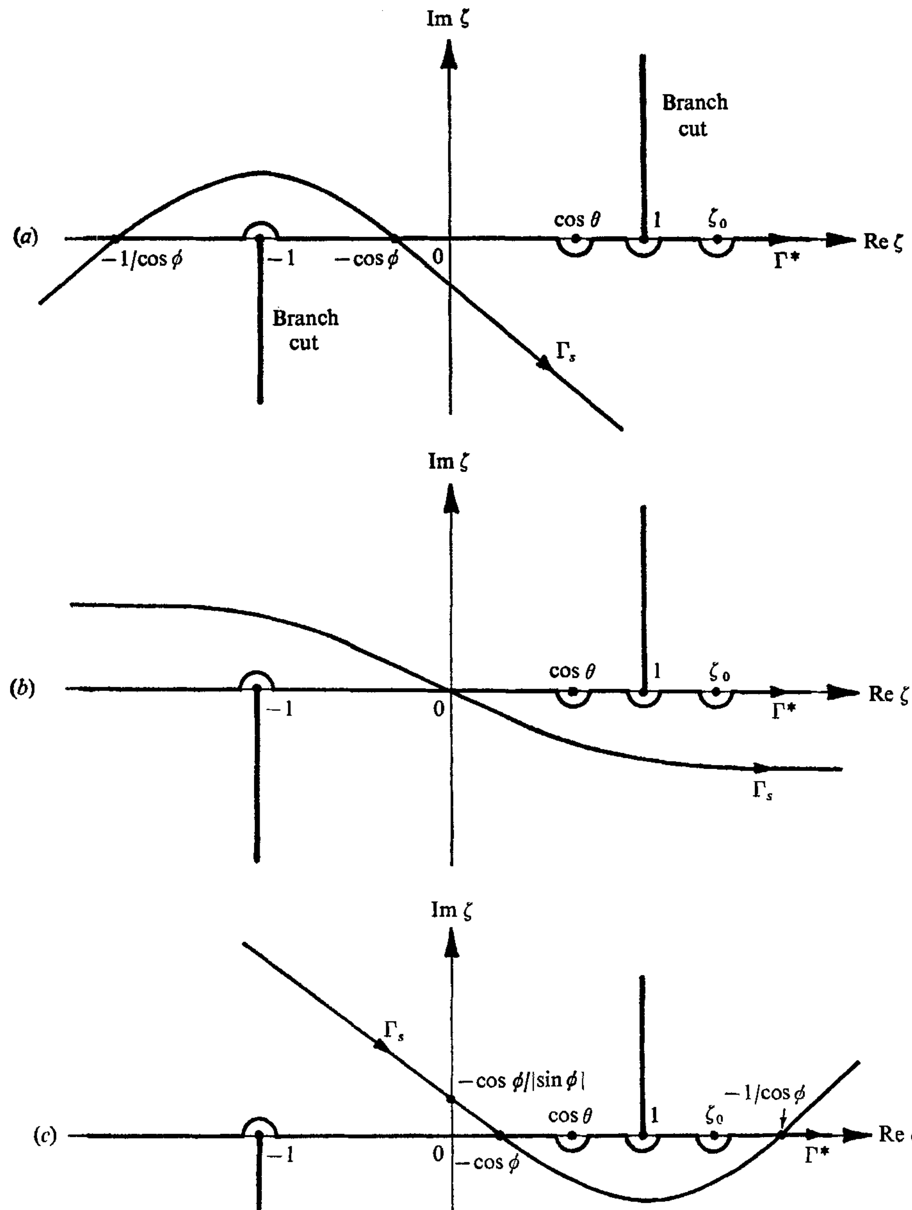

FIGURE 3. The ζ plane corresponding to (4.1) when $f < \sigma < N$ for the cases (a) $\cos\phi > 0$, (b) $\cos\phi = 0$ and (c) $\cos\phi < 0$. The path of steepest descent is denoted by Γ_s.

The semi-infinite barrier gives rise to a reflected free wave in the region II (figure 4); this is described by the third term in (4.3). Although the incoming and reflected waves are of the same amplitude, they are not in phase at the barrier unless either $\gamma = 0$ or $\cos\theta = 0$, i.e. unless there is no rotation or the incoming wave propagates normal to the barrier. This phase change is a consequence of

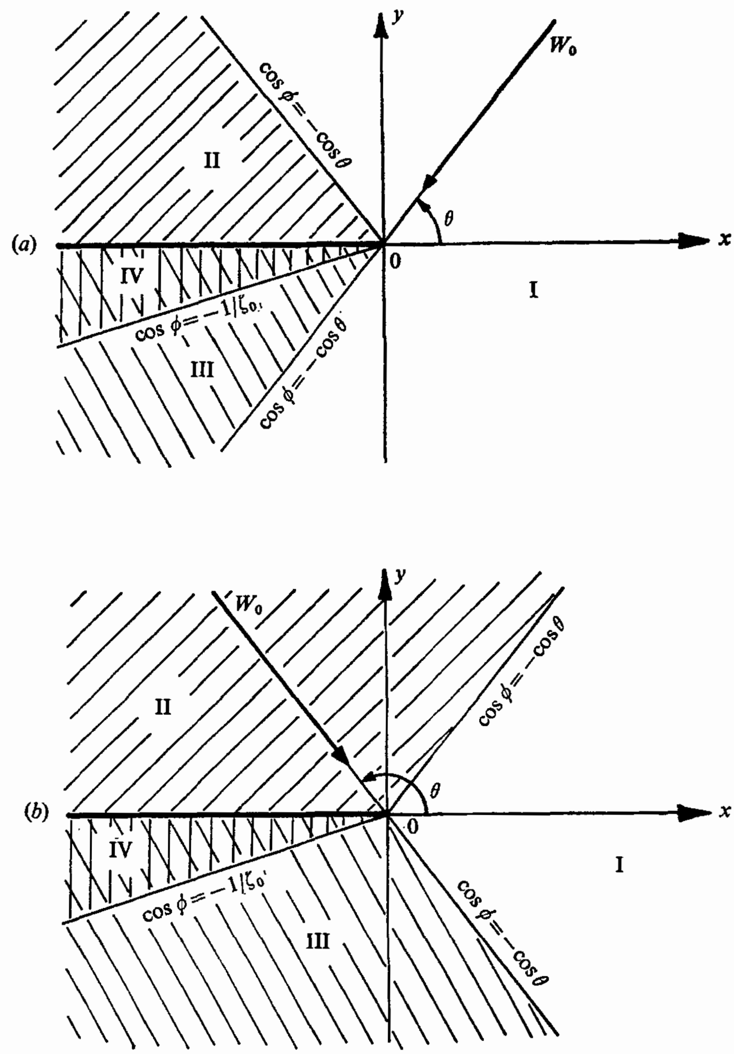

FIGURE 4. Regions around barrier in which terms of asymptotic solution (equation (4.3)) become important for the case $f < \sigma < N$ and where (*a*) $0 < \theta < \frac{1}{2}\pi$ and (*b*) $\frac{1}{2}\pi < \theta < \pi$.

the mixed boundary condition (2.13*b*). Behind the barrier, there is a 'shadow zone' (region III) in which no free wave exists. The last term in (4.3) describes an internal Kelvin wave propagating away from the origin with the barrier to its right ($f > 0$); i.e. it appears in region IV of figure 4. In figure 5, the amplitude of the Kelvin wave at the barrier is plotted as a function of γ for fixed values of θ. As indicated by Crease (1956), this wave can have a greater amplitude than that of the incoming wave. Because the amplitude is independent of k, it is also independent of N, the Brunt–Väisälä frequency. When $\gamma > 0$ and $\theta > 0$, as shown in figure 4, the Kelvin wave appears in the shadow zone behind the barrier. However, when one of the parameters γ and θ is negative, the wave exists on the same side of the barrier as the incoming wave.

5. Asymptotic solution for $N < \sigma < f$

For the passband $N < \sigma < f$, the waves described by (2.13 a, b) are not physically equivalent to long surface waves because in the latter case, $k^2 < 0$ when $\sigma < f$ and hence free waves of the type (2.11) do not exist. The waves of (2.13 a, b) are now strictly inertial internal waves.

It was shown in § 2 that to consider the diffraction problem for the case $N < \sigma < f$ which is physically similar to that for the case $f < \sigma < N$, we must replace k by $-k$ in (2.11)–(2.13). Now the Sommerfeld radiation condition is satisfied by assuming in (3.2) that k has a small negative imaginary part. Thus, the λ plane for this case is given by the reflexion of figure 2 across the $\operatorname{Im}\lambda$ axis, except that the poles at $\lambda = \pm\lambda_0$ are moved to $\lambda = \pm i|\lambda_0|$. The diffracted wave ψ is finally found to be given by (3.12) with k and λ_0 replaced by $-k$ and $ik(\gamma^2-1)^{-\frac{1}{2}}$, respectively; the contour Γ now passes below the singularities at $\lambda = -k$ and $-k\cos\theta$ and passes above the branch point at $\lambda = k$.

To determine the asymptotic form of ψ as $kr \to \infty$, we again apply the method of steepest descent. The saddle point is now located at $\zeta_s = \cos\phi$, and the possible paths of steepest descent are given by the reflexion of figure 3 about the $\operatorname{Im}\zeta$ axis, with the pole at ζ_0 moved to $i|\zeta_0|$. The contribution to ψ from the saddle-point is

$$\psi_s(r,\phi) = \frac{|\sin\phi|\,(i\sin\phi-\gamma\cos\phi)\,(\sin\theta-i\gamma\cos\theta)\,(1+\cos\theta)^{\frac{1}{2}}}{(\gamma^2-1)\,(i\zeta_1-\cos\theta)\,(i\zeta_1-\cos\phi)\,(\cos\phi+\cos\theta)\,(1-\cos\phi)^{\frac{1}{2}}} \times \frac{e^{-i(kr-\frac{1}{4}\pi)}}{(2\pi kr)^{\frac{1}{2}}} + O\left[\frac{1}{(kr)^{\frac{3}{2}}}\right], \quad (5.1)$$

where $\zeta_1 = (\gamma^2-1)^{-\frac{1}{2}}$. Thus, using Cauchy's theorem, the total wave mode ψ^T has the asymptotic form

$$\begin{aligned}\psi^T = {} & \psi_s + [H(\cos\phi+\cos\theta)+H(-\cos\phi-\cos\theta)\,H(\operatorname{sgn}\phi\operatorname{sgn}\theta)] \\ & \times \exp[ik(x\cos\theta+y\sin\theta)] \\ & + \frac{(\sin\theta-i\gamma\cos\theta)^2}{(\gamma^2\cos^2\theta+\sin^2\theta)}\,H(-\cos\phi-\cos\theta)\,H(\operatorname{sgn}\phi\operatorname{sgn}\theta)\exp[ik(x\cos\theta-y\sin\theta)] \\ & - \frac{2\gamma^{\frac{1}{2}}(1+\cos\theta)^{\frac{1}{2}}}{(\gamma^2-1)^{\frac{1}{4}}\,(\gamma\cos\theta-i\sin\theta)}\exp[i\tan^{-1}(\gamma^2-1)^{-\frac{1}{2}}]\,H(-y) \\ & \times H(-\cos\phi-(\gamma^2-1)^{-\frac{1}{2}}|\sin\phi|)\exp[(\gamma^2-1)^{-\frac{1}{2}}k(x+i\gamma y)], \quad (5.2)\end{aligned}$$

where ψ_s is given by (5.1). Clearly, the (x, y) plane may be divided as in figure 4, except that the Stokes line bordering region IV at $\cos\phi = -1/\zeta_0$ is replaced by one at $\cos\phi = -(\gamma^2-1)^{-\frac{1}{2}}|\sin\phi|$. The amplitudes of the waves in regions I to III are exactly the same as for the case $f < \sigma < N$; however, the phase of each wave is now propagating in the opposite direction. The last term in (5.2) represents an inertial-internal wave propagating in the region IV away from and normal to the barrier and decaying in the negative x direction. The maximum amplitude of this wave is shown as a function of γ for fixed values of θ in figure 5; as for the case $f < \sigma < N$, it can also be greater than the amplitude of the incoming wave. However, since these waves only appear at large distances from the origin (i.e. large kr), their actual amplitude is somewhat less than that shown in figure 5. On the other hand, since the ratio of the decay length to the wavelength is γ, the attenuation in the negative x direction is small over one wavelength for very low frequencies (i.e. for $\sigma \ll f$).

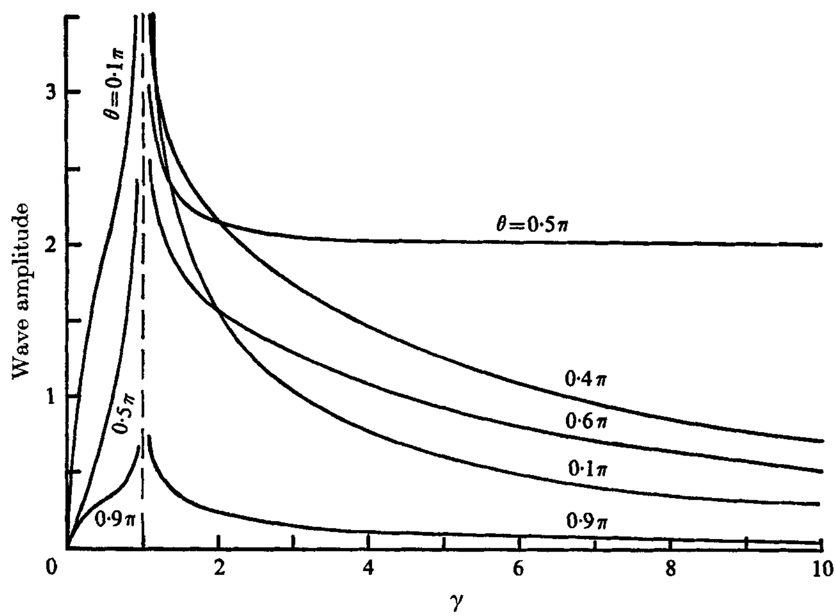

FIGURE 5. Maximum amplitude of internal Kelvin wave as a function of $\gamma = f/\sigma$ for various values of θ.

6. Discussion of internal Kelvin waves

The existence of internal waves of Kelvin type in region IV of figure 4 is understood by considering normal mode solutions of (2.9) and (2.10). These equations admit solutions of the form

$$W = e^{ilx+my} \sin(n\pi z/h), \tag{6.1}$$

where $l^2 = k^2/(1-\gamma^2) = (n\pi/h)^2 \sigma^2/(N^2-\sigma^2)$ and $m = -\gamma l$. The comparable normal mode long surface gravity wave problem (see §2) has the solution

$$\psi = e^{ilx+my}, \tag{6.2}$$

where $l^2 = k^2/(1-\gamma^2) = \sigma^2/gh > 0$ and $m = -\gamma l$. Equation (6.2) represents the familiar Kelvin-wave solution in which the wave is trapped against the barrier; i.e. the wave travels in the x direction and its amplitude decays exponentially away from the barrier. From (6.1), it is seen that the internal wave analogue of this wave occurs when $\sigma < N$, i.e. when $l^2 > 0$. Applying the radiation condition that energy must be propagating away from the origin, we find that for $\gamma > 0$ Kelvin waves can exist in the third quadrant only. Since they do not satisfy a continuity condition at the plane $x = 0$, they cannot exist alone. In fact, free waves present along the plane $x = 0$ $(y < 0)$ should tend to excite Kelvin waves which are 'natural oscillations' in this region. If the semi-infinite barrier is replaced by an infinite one, then Kelvin waves are exact solutions to the equations of motion. The wave-numbers of both types of Kelvin waves are independent of γ and the ratio of the decay length to the wavelength is γ^{-1} in each case. The waves can exist for all $\gamma(\neq 1)$, but they co-exist with free waves only when $\gamma < 1$, i.e. when $k^2 > 0$. We note that whereas long surface gravity Kelvin waves are non-dispersive, internal Kelvin waves are dispersive.

When $\sigma > N$, equation (6.1) describes a wave for which $l^2 < 0$. Thus, this inertial-internal wave propagates normal to the barrier and its amplitude decays along the barrier. As before, the radiation condition implies that the wave can exist only in the third quadrant of the (x, y) plane. However, because the amplitude of this wave increases exponentially in the x direction, it cannot exist alone, even if the barrier spans the whole x axis. The dispersion relation for the wave is, from (6.1),

$$m^2 = (n\pi/h)^2 f^2/(\sigma^2 - N^2).$$

The wave is clearly dispersive, and it is essentially an inertial wave because N may be set equal to zero without affecting the mathematics (or the physics) of the problem. It can also be shown that the phase and group velocities are of opposite signs. This wave in region IV described in §5 might thus be called a degenerate internal Kelvin wave or an inertial Kelvin wave.

We finally note that although there is an energy flux $(\frac{1}{2}\,\mathrm{Re}\,(pu^*))$ along the barrier associated with a steady-state internal Kelvin wave $(f < \sigma < N)$, there is no energy flux associated with a (steady-state) inertial Kelvin wave $(N < \sigma < f)$. On the other hand, by allowing the wave amplitude to vary slowly in time (i.e. by allowing the frequency σ to have a small positive imaginary part, ϵ), it is found that an inertial Kelvin wave is maintained by an energy flux along the barrier that is proportional to ϵ.

This work was supported in part by the Canadian National Research Council through operating grant A 5201.

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